Introduction to Gromov hyperbolicity of the curve complex

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Abstract

This paper serves as an introduction to the proof, improvement and applications of the Gromov hyperbolicity of the curve complex. The curve complex of a surface is a simplicial complex encoding the structure of the set of isotopy classes of essential simple closed curves, first introduced by Harvey. It is a canonical result of Masur and Minsky, that the curve complex is hyperbolic in the sense of Gromov. They gave a proof using the geometry of Teichmüller space and the theory of nested train-tracks. Later Bowditch gave a more combinatorial proof and showed that the hyperbolicity constant for the 1-skeleton of the curve complex is bounded by a logarithmic function. Furthermore, various authors showed that the hyperbolicity constant actually has a uniform bound independent of the surface itself, and some explicit bounds were given by Bowditch and Hensel, Przytycki, and Webb. The hyperbolicity of the curve complex directly leads to the relatively hyperbolicity of the Teichmüller space and the mapping class group. Besides, it has more applications to the group theoretic properties of the mapping class groups, thanks to more tools are developed to overwhelm the hardness due to locall infiniteness of the curve complex.

Contents

1 Introduction

The curve complex S of a surface S is a simplicial complex whose vertices are isotopy classes of essential simple closed curves on S, with $k-$simplices spanned by $(k+1)$-vertices which can be realized disjointly. It was first introduced by Harvey[11] and became an important tool in the study of Teichmuller spaces, mapping class groups, Heegaard splittings, etc. It is not hard to see that the curve complex is finite-dimensional, but locally infinite. One important property of the curve complex, discovered by Masur and Minsky, is that endowed with a natural metric, the curve complex is hyperbolic in the sense of Gromov, with infinite diameter. Their proof is a canonical one using Teichmüller geometry and treatment of train-tracks, but is not combinatorial in nature and lacks an estimate of the hyperbolicity constant. Nevertheless the result still has a great impact. It helps explain why the Teichmüller space has some negative-curvature properties in spite of not being itself hyperbolic, since the complex of curves exactly encodes the intersection patterns of the family of regions causing the failure of hyperbolicity. As a consequence, Teichmüller space is ”relatively hyperbolic” with respect to this family. This result can also be appied to mapping class groups to give various properties, such as relatively hyperbolicity with respect to a family of subgroups which leads to the failure of hyperbolicity of the mapping class group itself. There are subsequent improvement of this result, on the one hand, a purely-combinatorial proof is given by Bowditch [4], on the other hand, estimate on the hyperbolicity constant has been studied by Bowditch[4][5], Hensel, Przytycki, and Webb[19], Clay, Rafi, and Schleimer[17]. Besides, deeper applications to mapping class groups are refered to [9],[6].

The organazation of this paper is as follows.

Section 1 is an introduction to the background knowledge in the curve comlex, Gromov hyperbolicity for metric spaces and groups, and Teichmüller space along with its geometry.

Section 2 is a sketch of Masur and Minsky’s proof of the Gromov hyperbolicity of the curve complex, where I divide the proof into 4 steps and give descriptions respectively. Notice that this proof relies on the geometry of the Teichmüller space and do not give an estimate of the hyperbolic contant.

Section 3 introduces some improvements on Masur and Minsky’s proof. A proof using only combinatorial language was given by Bowditch, who also showed that there is a logarithmic function bounding the hyperbolicity constant, depending on the topological characteristic of the surface. Later, Bowditch[5], Hensel, Przytycki, and Webb[19], Clay, Rafi, and Schleimer[17] independently proved that there is a uniform upper bound for the hyperbolicity constant indepent of the surface. As for explicit bounds, for instance, Bowditch showed that if $2g+p \geq 107$, then the curve complex of the surface of genus g with p punctures is 1780-hyperbolic; Hensel, Przytycki, and Webb gave the constant 7 for general surfaces, but in another definition of hyperbolicity.

Section 4 shows some applications of the Gromov hyperbolicity of the curve complex to Teichmüller spaces and mapping class groups. Some direct applications are: the Teichmüller space is relatively hyperbolic with respect to a family of regions; the mapping class group is hyperbolic relative to a family of abelian subgroups. Then, although the curve comlex is locally infinite, with new tools developed by Masur and Minsky [9], by Dahmani, Guirardel, and Osin[6], more prolems in mapping class groups can be solved. Masur and Minsky introduced the notion of a hierarchy of tighted geodesics, and solved the conjugacy problem for mapping class groups. Dahmani, Guirardel and Osin developed ways through hyperbolically embedding and through very rotating families, and solved two open problems about mapping class groups.

2 Preliminaries

2.1 Curve complex

The concept of the curve complex was first introduced by Harvey in 1978 at the Riemann surfaces conference, whose content can be found in [11]. The main motivation is to provide an appropriate combinatorial framework for studying the geometry of how the moduler group $\Gamma(S)$ of a surface S acts at infinity on the Teichmüller space $T(S)$. Later, the curve complex turned out to be a fundamental tool in the study of the geometry of the Teichmüller space, of mapping class groups and of Kleinian groups. The following is Harvey's definition.

Definition 2.1 (Harvey)**.** Let S be an oriented surface of finite type, which means that its fundamental group is finitely generated. The curve complex $C(S)$ is a simplicial complex encoding the structure of the set of homotopy classes of simple closed curves on S, defined as follows:

- A vertice of C(S) is the free homotopy class of an essential (neither homotopically trivial or peripheral) simple closed curve on S.
- A k-simplex of C(S) is spanned by a (k+1)-tuple $\{\gamma_0, \gamma_1, \cdots, \gamma_k\}$ of distinct vertices which can be homotoped to be pariwise disjoint.

□

There are several well-known properties about the curve complex listed next.

Proposition 2.2 (Harvey)**.** *Let S be an oriented surface of genus g with p punctures. $C(S)$is a thick chamber complex of dimension $3g+p-4$, that's to say every simplex is a face of some $(3g+p-4)$-simplex and every $(3g+p-5)$ abutts at least three $(3g+p-4)$-simplices. Futhermore, C(S) is locally infinite.*

A result of Harer asserts that $C(S)$ is homotopically equivalent to a wedge sum of spheres in Theorem 2.1 in [10], where he used it as a tool for understanding mapping class groups of surfaces.

There is also an important property of the curve complex first stated by Harvey in [11] and essentially proved by Lickorish in [16] and this property was canonically used to prove the Dehn-Lickorish theorem which revealed a finite generating set of the mapping class group.

Proposition 2.3 (Harvay, Lickorish)**.** *Let S be an oriented surface of genus g with p punctures. If $3g+p \geq 5$, then $C(S)$ is connected.*

Let $C_k(S)$ be the k-skeleton of $C(S)$. For $C_1(S)$, there is a canonical combinatorial distance function d_C by specifying every edge has length 1: $d_C(\alpha,\beta) = \inf\{\text{length of } l : l \text{ is a path connecting } \alpha \text{ and } \beta\}, \forall \alpha,\beta \in C_0(S)$. Notice that in this way $C_1(S)$ becomes a geodesic metric space.

The distance in $C_1(S)$ is found in a way related to the geometric intersection pairing in S, according to Lemma 2.1 in [13]:

Proposition 2.4 (Masur, Minsky)**.** *For $\alpha,\beta \in C_0(S)$, let $i(\alpha,\beta)$ be the geometric intersection number of α with β on S, which is the smallest number of intersections of two curves in the free homotopy classes. For any $\alpha,\beta \in C_0(S)$, $d_C(\alpha,\beta) \leq 2i(\alpha,\beta)+1$.*

2.2 Gromov hyperbolicity

The following definitions of hyperbolicity, word hyperbolicity and relatively hyperbolicity were introduced and developed by Mikhail Gromov in [8], which is the reason why nowadays we call a metric space or a group hyperbolic in the sense of Gromov.

Definition 2.5 (Gromov)**.** X is a metric space. If some $x_0 \in X$ is chosen as a reference point, then we set $|x| = |x|_{x_0} = |x - x_0|$ and $(x.y) = (x.y)_{x_0} = 1/2(|x| + |y| - |x - y|)$. Call X hyperbolic with respect to x_0 if it satisfies the $\delta-$inequality: $(x.y) \geq min((x.z),(y.z)) - \delta$ for a fixed $\delta \geq 0$ and all x, y, z in X. X is called $\delta - hyperbolic$ if it is δ-hyperbolic with respect to each point $x \in X$, and we call X hyperbolic if it is δ-hyperbolic for some $\delta \geq 0$. □

Remark 2.6. If X is a geodesic metric space, the geodesic connecting any two points x, y in X is denoted as $[xy]$. The condition for X to be $\delta-$ hyperbolic is equivalent to the thin-triangle condition: there exists some $\delta \geq 0$ such that for any $x, y, z \in X$, the geodesic [xz] is contained in a δ-neighborhood of $[xy] \cup [yz]$. □

Definition 2.7 (Gromov)**.** Let Γ be an abstract group, and let a subset $G \subset \Gamma$ generate Γ, then there the word metric $|\cdot|_G$ on Γ is defined to be the maximal metric on Γ satisfying $|g| = |g^{-1}| = 1$ for every $x \in G$. Call Γ hyperbolic if it is hyperbolic as a metric space. If Γ is finitely generated, Γ is called word hyperbolic if a word metric for some finite generating subset in Γ is hyperbolic. □

Remark 2.8. The definition of the word hyperbolicity of a finitely generated group actually means that the Caylay graph of this group with respect to some finite generating set endowed with the graph metric is $\delta-$ hyperbolic for some $\delta \geq 0$. We notice that the word hyperbolicity is independent of the choice of a finite generating set, but the constant δ may depend on it, so we usually do not speak of Γ being δ-hyperbolic. □

The concept of relatively hyperbolicity is generalized from hyperbolicity, first introduced by Gromov in Section 8.6 of [8]. Basically, it generalizes the notion of a hyperbolic group to that of a group hyperbolic relative to a preferred class of "peripheral subgroups". After Gromov put forward this concept, relatively hyperbolicity has become a natural feature in the study of geometric group theory, and later researchers have given different points of view on the notion of relatively hyperbolicity (see [7],[20], [3]). Varies definitions take advantages in different contexts. Below is Gromov's initial definition using the geometric language of manifolds with cusps.

Definition 2.9 (Gromov)**.** Let X be a complete hyperbolic locally compact geodesic space with a discrete isometric action of a group Γ such that the quotient space $V = X/\Gamma$ is quasi-isometric to the union of k copies of $[0, +\infty)$ joined at 0. Suppose the action of Γ on X is free and the k rays in V are lifted to k rays in X: $\gamma_i : [0, +\infty) \to X, i = 1, \cdots, k$. Denote by h_i the corresponding (ray) horofunctions and denote by $\gamma_i(\infty) \in \partial X$ the limit points of γ_i. Denote by $\Gamma_i \subset \Gamma$ the isotropy group of $\gamma_i(\infty)$ for the action of Γ on ∂X and assume Γ_i preserves h_i. Denote by $B_i(\rho)$ the horoballs $h_i^{-1}(-\infty, \rho) \subset X$ and assume that for a sufficiently small ρ, the intersection $\gamma B_i(\rho) \cap B_j(\rho)$ is empty unless $i = j$ and $\gamma \in \Gamma_i$. Denote by $\Gamma B(\rho) \subset X$ the union $\cup_{i,\rho} \gamma B_i(\rho)$ over $i = 1, \cdots, k$ and all $\gamma \in \Gamma$. Set $X(\rho) = X \backslash \Gamma B(\rho)$, then we assume that the action of Γ on $X(\rho)$ is cocompact for all $\rho \in (-\infty, +\infty)$. Then we call Γ is hyperbolic relative to subgroups $\Gamma_1, \cdots, \Gamma_k$ in Γ. □

One canonical example Gromov gave is a finite volume discrete isometry group Γ of a complete simply connected Riemannian manifold X with pinched negative curvature $0 > -a > K(X) > -b >$

$-\infty$. Knowing X/Γ is quasi-isometric to the wedge of several copies of $[0,+\infty)$, Γ is hyperbolic relative to the isotropy subgroups of $\gamma_i(\infty) \in \partial X$. Actually, fundamental groups of of complex hyperbolic manifolds with cusps are not word-hyperbolic, since they do not exhibit nonpositively curved geometry but exhibit a nontrivial mix of both negatively curved and nilpotent geometry ([7]). Thus relatively hyperbolicity is a tool to exploit the negatively curved part by considering the relation between the fundamental groups of cusps with the whole fundamental group.

This initial deifinition can be simplified to the equivalent definition below, which is Definition 1 of [3] stated by Bowditch.

Definition 2.10 (Bowditch)**.** We say a group Γ is hyperbolic relative to a set $\mathscr{G}$ of subgroups, if Γ admits a properly discontinuous isometric action on a path-metric space X, with the following properties:

(1) X is proper (i.e. complete and locally compact) and hyperbolic,

(2) every point of the boundary of X is either a conial limit point or a bounded parabolic point,

(3) the elements of $\mathscr{G}$ are precisely the maximal parabolic subgroups of Γ,

(4) every element of $\mathscr{G}$ is finitely generated.

$\square$

Bowditch aslo gave another equivalent definition shown below and gave a proof of the equivalence in [3]. The former is viewd as a dynamical characterization, and the latter is phrased in terms of group actions on sets.

Definition 2.11 (Bowditch)**.** We say a group Γ is hyperbolic relative to a set $\mathscr{G}$ of subgroups, if Γ admits an action on a connected graph K, with the following properties:

1. K is hyperbolic, and each edge of K is contained in only finitely many circuits of length n for any given integer n,
2. there are finitely many Γ-orbits of edges, and each edge stabilizer is finite,
3. the elements of $\mathscr{G}$ are precisely the infinite vertex stabilizers of K,
4. every element of $\mathscr{G}$ is finitely generated.

$\square$

Farb gave another definition by confining the properties of the modification of the Caylay graph in [7], while this definition turns out to be weaker than the original definition given by Gromov, with a specific example pointed out by Szczepanski in [20].

Definition 2.12 (Farb)**.** There are two definitions of relatively hyperbolicity in geodesic metric spaces relative to family of regions and groups relative to subgroups, respectively:

- Let X be a geodesic metric space, and let $\mathscr{H}$ be a family of regions. The electric distance d_e on X is the path metric imposed by shrinking each $H \in \mathscr{H}$ to diameter 1 in the following way: For each $H \in \mathscr{H}$ create a new point c_H and an interval of length $1/2$ from c_H to every point in H. The enlarged space $\hat{X}$ is called the electric space, with the new metric induced by shortese paths. We say X is hyperbolic relative to $\mathscr{H}$ if $(\hat{X}, d_e)$ is δ-hyperbolic for some $\delta \geq 0$.

- Let G be a fintely generated group, and let $\{H_1, \cdots, H_r\}$ be a finite set of finitely generated subgroups of G. Γ is the Caylay graph of G, and a new graph $\hat{\Gamma} = \hat{\Gamma}(\{H_1, \cdots, H_r\})$ called the coned-off Caylay grach with respect to $\{H_1, \cdots, H_r\}$ is constructed as follows: for each coset $gH_i (1 \leq i \leq r)$ of H_i in G, add a vertex $v(gH_i)$ to Γ and add an edge $e(gh_i)$ of length $1/2$ from each element gh_i of gH_i to the vertex $v(gH_i)$. The resulting $\hat{\Gamma}$ is also a geodesic metric space. We say G is hyperbolic relative to $\{H_1, \cdots, H_r\}$ if the coned-off Caylay gragh $\hat{\Gamma}$ of G with respect to $\{H_1, \cdots, H_r\}$ is a negatively curved metric space.

$\square$

The main idea of why the definition of Gromov is stronger than that of Farb was proposed by Bowditch, who proposed that the surgeries on a δ-hyperbolic metric space by contracting certain quasiconvex subsets properly retain the property of being hyperbolic. The example Szczepanski gave is that for the group $G = \mathbb{Z} \otimes \mathbb{Z}$ and its subgroup $H = \mathbb{Z}$, G is hyperbolic relative to H in the sense of Farb but not in the sence of Gromov, since any $\mathbb{Z} \otimes \mathbb{Z}$ acting on a hyperbolic space has a fixed point. Notice that in [13], the notion of relatively hyperbolicity is in the sense of Farb.

2.3 Teichmüller space

Here I give a brief introduction to the Teichmüller space and its geometry, mainly due to its close relation with the the curve complex, as well as its efficiency in the proof of Gromov hyperbolicity of the curve complex.

The Teichmüller space was first defined and studied by Fricke, Teichmüller, Fenchel, and Nielson. In most context, the Teichmüller space $T(S)$ is defined to be the isotopy classes of complex structures on an orientable smooth surface S. Specifically, a complex structure on S means a diffeomorphism $\phi : S \to X$, where X is a Riemann surface; two complex structures $\phi_1 : S \to X_1$ and $\phi_2 : S \to X_2$ are isotopic if there is an isometry $I : X_1 \to X_2$ such that $I \circ \phi_1$ and ϕ_2 are isotopic. Teichmüller spaces can also be defined equivalently using conformal structures. Furthermore, for a compact orientable surface with finite puntures and negative Euler characteristic, the Teichmüller spaces can also be defined using hyperbolic structures, e.g. in [2].

There is a canonical topology on the Teichmüller space induced by length functions from the set $\mathscr{S}$ of isotopy classes of essential simple closed curves to $\mathbb{R}_+$. For any $\chi \in T(S)$ representing the isotopy class of some complex structure $\phi : S \to X$, the corresponding length funtion $l_\chi : \mathscr{S} \to \mathbb{R}_+$ is defined by assigning $\alpha \in \mathscr{S}$ the length of the unique geodesic in X in the isotopy class $\phi(\alpha)$. Then the map $l : T(S) \to \mathbb{R}^{\mathscr{S}}, \chi \mapsto (l_\chi(\alpha))_{\alpha \in \mathscr{S}}$ turns out to be injective, which endows $T(S)$ with the subspace topology, now that $\mathbb{R}^{\mathscr{S}}$ has the product topology. There follows the problem of counting its dimension, and one fact is that the Teichmüller space of a closed surface of genus g denoted as S_g is homeomorphic to $\mathbb{R}^{6g-6}$, proved by Teichmuller, and there is also a famous relevant theorem called the $9g-9$ theorem telling us that there is an embedding of $T(S_g)$ to $\mathbb{R}^{9g-9}$. More details and relevant proofs can be found in Chapter 10 of [2].

The geometry of Teichmüller is of more concern in this article. I will review some basic concepts and well-known results about the Teichmüller geometry which are useful in the proof of Gromov hyperbolicity of the curve complex, as well as in the applications of the result for the curve complex to the Teichmüller space.

Definition 2.13. Suppose that f is a (not necessarily continuous) function from one metric space (M_1, d_1) to a second metric space (M_2, d_2). Then f is called a quasi-isometry from (M_1, d_1) to

(M_2, d_2) if there exist constants $A \geq 1, B \geq 0, C \geq 0$ such that the following two properties both hold:

- For every two points x and y in M_1, the distance between their images is controlled by their original distance with respect to constants A, B. More formally: $\forall x, y \in M_1, \frac{1}{A}\, d_1(x,y) - B \leq d_2(f(x), f(y)) \leq A\, d_1(x,y) + B$.
- Every point of M_2 is within the constant distance C of an image point. More formally: $\forall z \in M_2, \exists x \in M_1, s.t.\ d_2(z, f(x)) \leq C$.

The two metric spaces (M_1, d_1) and (M_2, d_2) are called quasi-isometric if there exists a quasi-isometry f from (M_1, d_1) to (M_2, d_2).

□

Definition 2.14. Let $f : X \to Y$ be an orientaion-preserving homeomorphism between Riemann surfaces that is smooth outside of a finite number of points. The dilatation of f at $p \in X$ is defined to be $K_f(p) = \frac{|f_z(p)| + |f_{\bar{z}}(p)|}{|f_z(p)| - |f_{\bar{z}}(p)|}$, and the dilatation of f is defined to be $K_f = \sup K_f(p)$, where p ranges over all points where f is differentiable. If $K_f < \infty$, we say that f is quasiconformal or K_f-quasiconformal. □

Definition 2.15. Let X be a Riemann surface, a holomorphic quadratic differential on X is a holomorphic section of the symmetric square of the holomorphic contangent boundle of X. In terms of local coordinates, after taking an atlas $\{z_\alpha : U_\alpha \to \mathbb{C}\}$ of X, a holomorphic quadratic differential q on X is specified by a collection of expressions $\{\phi(z_\alpha) dz_\alpha^2\}$ with the following properties:

- Each $\phi_\alpha : z_\alpha(U_\alpha) \to \mathbb{C}$ is a holomorphic function with a finite set of zeros.
- For any two coordinate charts z_α and z_β, we have $\phi_\beta(z_\beta)(\frac{dz_\beta}{dz_\alpha})^2 = \phi_\alpha(z_\alpha)$.

The set of all holomorphic quadratic differentials on X forms a complex vector space denoted as $QD(X)$. □

Problem arose about whether there exists a unique quasiconformal homeomorphism between two Riemann surfaces in any given homotopy class, realizing the minimal dilatation. This problem is usually called Teichmüller's extermal problem, considered by Grotzsh first in the case of rectangles then by Teichmüller for general Riemann surfaces. This problem has a positive solution stated in next theorem called Teichmüller's theorem, first given by Teichmüller and Ahlfors. The map realizing the minimal dilatation is called a Teichmüller map, which has a neet form in local charts.

Theorem 2.16 (Teichmüller's theorem)**.** *Let X and Y be two closed Riemann surfaces of genus g. We call a homeomorphism $f : X \to Y$ a Teichmüller map if there are holomorphic quadratic differentials q_X and q_Y on X and Y respectively, and a positive real number K such that: on the one hand, f takes the zeros of q_X to the zeros of q_Y; on the other hand, if $p \in X$ is not a zero of q_X, then in the natural coordinates at p for q_X and $f(p)$ for q_Y, $f(x+iy) = \sqrt{K}x + i\frac{1}{\sqrt{K}}y$. The Teichmüller map satisfies the following two properties:*

- *(exsitence) Let X and Y be closed Riemann surfaces of genus $g \geq 1$, and let $f : X \to Y$ be a homeomorphism. Then there exists a Teichmüller map $h : X \to Y$ homotopic to f.*

- *(uniquenuess) Let $h : X \to Y$ be a Teichmüller map between two closed Riemann surfaces of genus $g \geq 1$. If $f : X \to Y$ is a quasiformal homeomorphism homotopic to h, then $K_f \geq K_h$. Equality holds if and only if $f \circ h^{-1}$ is conformal. In particular, if $g \geq 2$, equality holds if and only if $f = h$.*

Thus we know in the case of $g \geq 2$, in a given homotopy class of quasiconformal homeomorphisms between Riemann surfaces, the Teichmüller map is the unique one realizing the minimal dilatation.

Proof. e.g. see Chapter 11 in [2]. □

Next, the Teichmüller metric is defined naturally due to above definitions and theorem. Proposition 11.17 of [2] also states that the Teichmüller metric is complete.

Definition 2.17. Let $x, y \in T(S)$ and suppose x is represented by $\phi_1 : S \to X$ and y is represented by $\phi_2 : S \to Y$. $f = \phi_2 \circ \phi_1^{-1} : X \to Y$ is the change-of-marking map. There is a Teichmüller map $h : X \to Y$ in the homotopy class of f, guaranteed by Teichmüller's theorem. Then the Teichmüller distance between x and y is defined to be: $d_T(x,y) = \frac{1}{2} log(K_h)$. □

While in [13], authors did not take this definition of the Teichmüller metric, but used Kerckhoff's result of a equivalent definition using the notion of the extermal length. The extermal length is a conformal invariant of a isotopy class of simple closed curves which was introduced by Beurling and developed by Ahlfors and him, whose definition has both an analytic version and a geometric version (see e.g. [15]):

Definition 2.18. The extermal length has two equivalent definitions below:

- (analytic) Given a Riemann surface S and $x \in T(S)$, for a conformal metric on (S, x) with local form $\rho(z)|dz|$ and a simple closed curve γ in S, let $l_\rho(\gamma)$ denote the infinum of lengths of simple closed curves isotopic to γ measured with respect to ρ, and let A_ρ denote the area of S with respect to ρ. The extermal length of γ in S is $Ext_x(\gamma) = sup_\rho \; l_\rho(\gamma)^2/A_\rho$, where ρ ranges over all conformal metrices with $0 < A_\rho < \infty$.
- (geometric) The extermal length $Ext_x(\gamma) = 1/mod(\gamma)$, where $mod(\gamma)$ is the supremum of the moduli of all cylinders embedded in S with core curve isotopic to γ.

□

The following is the result of Kerckhoff as to the Teichmüller metric, as Theorem 4 in [15].

Theorem 2.19 (Kerckhoff)**.** *The Teichmüller distance between two points x, y in $T(S)$ is equal to $1/2 \; log(sup_{\gamma \in C_0(S)} \frac{Ext_y(\gamma)}{Ext_x(\gamma)})$ where $Ext_x(\gamma)$ denotes the extermal length of γ in (S, x).*

Teichmüller space being a metric space, what are its geodesics like is of concern. In fact, geodesics in $T(S)$ are determined by holomorphic quadratic differentials. Recalling the definition of the Teichmüller map (Theorem 2.15) $f : X \to Y$, if we fix X and q_X but vary K in $(0, \infty)$, we obtain a one-parameter subset of $T(S)$, called a Teichmüller line and all Teichmüller lines account for all geodesics in $(T(S), d_T)$ (see e.g. [2]). In terms of horizontal and vertical lengths, a geodesic denoted as $L_q(t)$ parametrized by arclength can also be written in the following way. A point $L_q(t)$ in $T(S)$ is determined by its holomorphic quadratic differential q_t where for any closed curve or arc α in S, the horizontal length in the q_t metric is $|\alpha|_{q_t,h} = |\alpha|_{q_0,h} e^t$, and the vertical length $|\alpha|_{q_t,v} = |\alpha|_{q_0,v} e^{-t}$.

It's well known that the mapping class group $Mod(S)$ acts properly discontinuous on the Teichmüller space $T(S)$ and the quotient $M(S) = T(S)/Mod(S)$ is the moduli space of Riemnn surfaces, which is another important topic in geometric group theory. The mapping class group is oberseved not to be Gromov hyperbolic since it contains high rank abelian subgroups, while this does not directly imply that the Teichmüller space is not Gromov hyperbolic, since there are examples of a Gromov hyperbolic group acting on a Gromov hyperbolic space. Still, the Teichmüller space with the Teichmüller metric turns out not to be Gromov hyperbolic, due to the fact that a short curve α decomposes the surface into pieces that are geometrically nearly independent, giving the region $H_\alpha := \{\chi \in T(S) | Ext_\chi(\alpha) < \varepsilon_0\}$ for sufficiently small ε_0 the approximate geometry of a product, whereas a product of infinite-diameter spaces can not be Gromov hyperbolic. This is proved by Masur and Wolf in [12] stated as the next theorem.

Theorem 2.20 (Masur, Wolf). *Teichmüller space with the Teichmüller metric is not Grovmov hyperbolic.*

3 Curve complex is Gromov hyperbolic

In this section, I will focus on the main theorem in [13] about the Gromov hyperbolicity of the curve complex, and give a sketch of its proof given by Masur and Minsky.

The following theorem is mainly motivated by the attempt to explain the presence and absence of negative-curvature properties in the Teichmüller space. To be more specific, the Teichmüller space actually fails to be Gromov hyperbolic due to the presence of infinite diameter regions, whose intersection pattern is exactly encoded by the curve complex which is Gromov hyperbolic; thus if the internal structure of these regions is properly ignored, the negative-curved part of the Teichmüller space will appear, in other words, the Teichmuller space is hyperbolic relative to these regions. I will illustrate the relatively hyperbolicity of the Teichmuller space in Section 5 where I will talk about the applications of hyperbolicity of the curve complex. Besides, in a long term, this theorem plays an important role in benefiting the undertanding of the mapping class groups, as discussed in Section 5.

Theorem 3.1 (Marsur, Minsky). *Let S be an oriented surface of finite type. The curve complex $C(S)$ is a δ-hyperbolic metric space, where δ depends on S. C(S) has infinite diameter except when S is a sphere with 3 or fewer punctures.*

Sketch of proof. I will divide the proof of Masur and Minsky into several steps and try to give enough information for each step to assure that it is understandable.
Step 0: First of all, the surface S can be restricted to the case with $3g + p \geq 5$, where the 1-skeleton $C_1(S)$ is a geodesic metric space.

It is because the sporadic cases left are surfaces with $g = 0, p \leq 4$ and $g = 1, p \leq 1$, whose curve complexes are either trivial or well-understood. If S is with $g = 0, p \leq 3$, there are no essential curves, so its curve complex is empty. If S is with $g = 0, p = 4$ or $g = 1, p \leq 1$, there are not a pair of disjoint nonisotopic essential curves, so its curve complex has no edges but infinite vertices, in which case people ususally consider about the Farey complex whose edges are defined to have smalleset possible intersection number, not necessarily 0 (1 for tori and 2 for 4-times punctured sphere). The Farey complex of a torus or a once-punctured torus or a 4-times punctured sphere, turns out to be easily proved to be Gromov hyperbolic. Minsky gave a proof in Section 3 of [21]:

Take the torus as an example, other cases being similer. The Farey complex of the torus has a canonical way of embedding as an ideal triangulation in $\mathbb{H}^2$ shown in Figure 1: The 0-skeleton of the Farey complex is isomorphic $\mathbb{Q} \cup \{\infty\}$ embedded in the boundary circle; for the 1-skeleton, notice that the intersection number of curves represented by p/q and r/s is $|ps - qr|$, so if there is an edge connecting them, $|ps - qr| = 1$, and what's left are some easy computations.

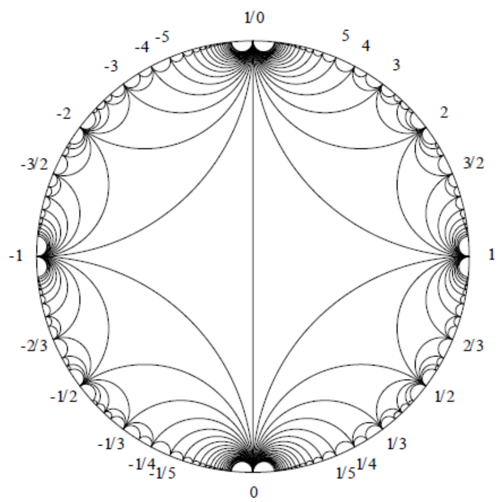

Figure 1: *The Farey complex of the torus*

Given two vertices x, y, let $E(x,y)$ be the set of all edges separating x and y into two sides. $E(x,y)$ admits an order where $e < f$ if e separates x from the interior of f, and a vertex shared in successive elements in this order is called a pivot. For $e \in E(x,y)$, there are two cases: if e is incident to no pivots, then $E(x,y) = \{e\}$ and x, y are opposite vertices of a quadrilateral consisting of two triangles, so $d(x,y) = 2$ and there are only two possible geodesics connecting them; if e is incident to a pivot p, by analyzing the configuration of the sequence of edges incident to p, there are only finite well-defined ways for a geodesic connecting x and y to pass through the block of edges around each pivot, and particularly they must pass entirely through the vertices, hence lying in the $1/2$-neighborhood of $E(x,y)$.

Now for a triple x, y, z, let $e \in E(x,z)$. Either e separates $\{x,y\}$ or $\{y,z\}$, i.e. in $E(x,y) \cup E(y,z)$, or y is a vertex of e. Thus $E(x,z) \subset E(x,y) \cup E(y,z) \cup \{\text{edges incident to y}\}$. Due to the fact that the distance of $[xy]$ is at most $1/2$ from $E(x,y)$, we deduce that $[xz]$ is at most distance $3/2$ from $[xy] \cup [yz]$. Thus the Farey complex is $3/2$-hyperbolic.

By the way, for this simple case, the Teichmüller space of the torus is isometric to the Poincaré disk, and for the torus, there exists a constant ε_1 such that there are at most two curves with length ε and their intersection number must be 1. Thus the Farey complex of the torus is exactly the nerve of the regions $T(\alpha) = \{\chi \in T(S) | l_\chi(\alpha) < \varepsilon_1\}$.

Step 1: The Teichmüller geodesics in the Teichmüller space induce a family of paths in the curve complex as follows:

First, there is a natural map $\Phi : T(S) \to \{U \subset C(S) | U \text{ is finite}\}$, mapping $x \in T(S)$ to the set of isotopy classes of curves realizing the shortest extermal length in (S,x). Let q be a quadratic differential on a Riemnn surface (S,x), and let $L_q(t) : \mathbb{R} \to T(S)$ be the corresponding Teichmüller geodesic. It induces a path in $C(S)$ denoted as $F_q : \mathbb{R} \to C(S)$, by maping t to one of the curves in $\Phi(L_q(t))$. By varying q, we get a family of paths in $C(S)$.

This family of paths in $C(S)$ satisfies two important properties which are crucial in the proof:
(i) coarsely transitive

Definition 3.2. Let X be a metric space. We say that a family Γ of paths is **coarsely transitive** if there exists a constant $D \geq 0$ such that for any $x, y \in X$ with $d(x, y) \geq D$, there exists $\gamma \in \Gamma$ joining x to y. □

The family $\{F_q\}$ of curves in $C(S)$ is coarsely transitive with the constant $D = 3$. For any $\alpha, \beta \in C(S)$ such that $d_C(\alpha, \beta) \geq 3$, there is no γ disjoint from both, so α and β fill S, that is, α and β can be realized in minimal position and the complement of $\alpha \cup \beta$ is a union of topological disks. Therefore there is a quadratic differential q with its nonsingular vertical leaves homotopic to α and horizontal ones homotopic to β, so F_q is a path connecting α and β, with $F_q(+\infty) = \alpha, F_q(-\infty) = \beta$.
(ii) satisfying the contraction property

Definition 3.3. Let X be a metric space. We say that a family Γ of paths has the **contraction property** if there exist uniform constants $a, b, c > 0$ such that for any path $\gamma : I \to X$ (where $I \subset \mathbb{R}$ is some interval, possibly infinite), there exists a projection $\pi : I \to X$ satisfying:

(1) For any $t \in I$, $diam(\gamma([t, \pi(\gamma(t))])) \leq c$.

(2) If $d(x, y) \leq 1$, then $diam(\gamma([\pi(x), \pi(y)])) \leq c$.

(3) If $d(x, \gamma(\pi(x))) \geq a$ and $d(x, y) \leq b \cdot d(x, \gamma(\pi(x)))$, then $diam(\gamma([\pi(x), \pi(y)])) \leq c$.

(Here for $s, t \in \mathbb{R}$, $[s, t]$ refers to the interval with endpoints s, t regardless of order.) □

The contraction property roughly says that, firstly, points in $\gamma(I)$ move slightly, secondly the projection π is coarsely Lipschitz, and lastly the map $\gamma \circ \pi$ is strongly contracting for points far away from their images in $\gamma(I)$ in the large.

The projection $\pi_q : C(S) \to \mathbb{R}$ correponding to $F_q : \mathbb{R} \to C(S)$ is defined in the following way. For $\alpha \in C(S)$, after we take its q-geodesic representative α^*, there are three cases: if α^* is vertical with respect to q, i.e. $|\alpha^*|_{q,h} = 0$, we define $\pi_q(\alpha) = +\infty$; if α^* is horizontal with respect to q, i.e.$|\alpha^*|_{q,v} = 0$, we define $\pi_q(\alpha) = -\infty$; if α^* is neither vertical or horizontal, i.e.$|\alpha^*|_{q,v} \neq 0$ & $|\alpha^*|_{q,h} \neq 0$, noticing that α^* is still a geodesic with respect to q_t and $|\alpha^*|_{q_t,v} = |\alpha^*|_{q,v}e^{-t}, |\alpha^*|_{q_t,h} = |\alpha^*|_{q,h}e^t$, there must be a unique t such that $|\alpha^*|_{q_t,v} = |\alpha^*|_{q_t,h}$ (in this case we call α^* balanced with respect to q_t), then we define $\pi_q(\alpha) = t$.

The consequence of the path family $\{F_q\}$ with the projections $\{\pi_q\}$ satisfying the contraction property is called the **"Projection Theorem"** as Theorem 2.6 in [13].

In order to prove the Projection Theorem, the authors took a long time establishing basic tools to control the distances between curves in $C_0(S)$. Section 3 of [13] gives a thorough introduction to the treatment of train-tracks, and refers to [18] for a complete treatment. I will give a brief introduction to it and quote crucial lemmas beneficial for the proof.

A train track on a surface S is a smoothly embedded 1-complex with edges (called branches) meeting mutually tangently at all vertices (called switches). The study of train tracks was originally motivated by the following observation: if a generic lamination (a partition of a closed subset of the surface into smooth curves) on a surface is looked at from a distance by a myopic person, it will look like a train track. A transverse measure on a train-track τ (also called weight) is a non-negative function μ on the branches satisfying the switch condition: for any switch, the sums of μ over incoming and outgoing branches are equal (”incoming” and ”outgoing” branches are distinguished by the directions of their tangent vector at the switch). A simple closed curve α is saied to be carried on τ if it is homotopic to nondegenerate smoothly immersed curve in τ, in which case the degree of the covering of each branch gives a transverse measure on τ. Thus in later context we regard an essential curve as a transverse measure on a train track. For better applications, it's important to constrain the topological shape of a train-track τ. On the one hand, τ is required to be recurrent, that is, every branch is contained in a closed train route, or equivalently there is a transverse measure positive on every branch. On the other hand, one can further require that τ be birecurrent which additionally requires τ to be transversely recurrent, namely every branch of τ is crossed by some simple curve intersecting τ transversely and efficiently (no bigon complementary components), or equivalently for any positive numbers L (large) and ε (small), there is a complete finite-area hyperbolic metric on S in which τ can be realized so that all edges have length at least L and curvature at most ε.

Below are some relevant notions.

Notions. Let τ, σ be train-tracks on S,

(1) $P(\tau)$: The polyhedron of measures supported on τ, seen as a subset of $ML(S)$ (the space of all compactly supported measured geodesic laminations on S) and a subset of the space $R_+^{\mathscr{B}}$ of non-negative functions on the branch set $\mathscr{B}$ of τ.

(2) $int(P(\tau)$: The set of weights on τ which are positive on every branch.

(3) $\sigma < \tau$: σ is a subtrack of τ, i.e. τ is a extension of σ (equivalently $P(\sigma)$ is a subsurface of $P(\tau)$).

(4) $\sigma \prec \tau$: σ is carried on τ, i.e. there is a homotopy of S taking every train route in σ to a train route in τ (equivalent to $P(\sigma) \subset P(\tau)$).

(5) σ fills τ: $\sigma \prec \tau$ and $int(P(\sigma)) \subset int(P(\tau)$ (especially, if τ and σ are recurrent, every branch of τ is transversed by some branch of σ).

(6) Call τ or $P(\tau)$ large if all the components of $S \backslash \tau$ are polygons or once-punctured polygons.

(7) Call τ maximal if it's not a proper subtrack of any other track, i.e. all complementary regions of τ are triangles or puntured monogons (especially, except in the case of punctured torus, it's equivalent to $dim(P(\tau)) = dim(ML(S))$.

(7) A diagonal extension of a large track τ is a track k such that $\tau < k$ and every branch of $k \backslash \tau$ is a diagonal of τ. If τ is transversely recurrent, so is any diagonal extension.

(8) $E(\tau)$ =the set of all recurrent diagonal extensions of τ, $PE(\tau) = \cup_{k \in E(\tau)} P(k)$.

(9) $N(\tau)$ =the union of $E(\sigma)$ over all large recurrent subtracks $\sigma < \tau$, $PN(\tau) = \cup_{k \in N(\tau)} P(k)$.

(10) $int(PE(\tau))$ =the set of measures $\mu \in PE(\tau)$ which are positive on every branch of τ, $int(PN(\tau)) = \cup_k int(PE(k))$ where k varies over large recurrent subtracks of τ.

An obeservation. If α, β are disjoint curves and α is carried on a maximal train-track σ in such a way that it passes through every branch, then β is also carried on σ. In other words, $\mathscr{N}_1(int(P(\sigma)) \subset P(\sigma)$, where $\mathscr{N}_1$ denotes the 1-neighhorhood in $C_1(S)$. This fact can be generalized to the case for diagonal extensions as follows.

Lemma 3.4. *If σ is a large bicurrent train-track and $\alpha \in int(PE(\sigma))$, if $d_C(\alpha, \beta) \leq 1$, then $\beta \in PE(\sigma)$. That is, $\mathscr{N}_1(int(PE(\sigma))) \subset PE(\sigma)$.*

It also induces the case for large subtracks:

Corollary 3.5. $\mathscr{N}_1(int(PN(\sigma))) \subset PN(\sigma)$.

Its partial converse is the following Nesting lemma:

Lemma 3.6 (Nesting lemma)**.** *There exists a $D_2 > 0$ such that whenever w and τ are large generic tracks and $w \prec \tau$, if $d_C(w,\tau) \geq D_2$, we have $PN(w) \subset int(PN(\tau))$.*

Back to our goal to verify the contraction property, first we need to prove, if α is a point of $\Phi(L_q(t))$ and β, γ are two curves in $C_0(S)$ which are far away from α but relatively close to each other, and suppose β projects to α, then we have the projection of γ is at a bounded distance from α. This is obtained by examing translations along L_q from mostly vertical to balanced to mostly horizontal curves. Consider a very long nearly vertical segment with respect to q_0, which does not fill the whole surface (say it avoids a definite-length horizontal segment), then if for $t > 0$ the segment is still long and nearly vertical, it fills up some proper subsurfaces which can only shrink as t increases, or specifically the subsurface filled by the vertical part steadily makes a bounded number of topological reductions until it is a disk. The boundaries of the resulting sequences of surfaces form a bounded-length sequence in $C(S)$. Using this argument, the authors show that if β is balanced and γ is far from balanced, say more vertical than horizontal, it takes a bounded distance to a point where β is almost completely horizontal and γ is still almost completely vertical and both fill up the whole surface.

At this point, a contradiction is derived using the train-track machinery. The distance of β from α implies the existence of a long sequence of nested train-track polyhedra which contain β and the bound on $d(\beta,\gamma)$ then traps γ deep inside a long subsequence of these polyhedra. It follows that there is a train-track which simultaneously carries both β and γ, but intersects any short curve on the surface many times. Particularly the intersection number of β and γ with each other is considerably smaller than their intersection numbers with any short curve. This contradicts the fact that one of them is nearly horizontal and the other is nearly vertical. Thus in fact β and γ are balanced at a bounded distance apart in $\Phi(L_q)$. What's left is checking the conditions for the contraction property one by one using the above observations.

Step 2: There is a probably well-known theorem asserting that a metric space having a coarsely transitive path family with the contraction property is Gromov hyperbolic (Theorem 2.3 of [13]). With this theorem, the curve complex turns out to be Gromov hyperbolic.

The proof of this theorem is divided into two lemmas below. Some concepts are defined first.

Definition 3.7. Let X be a geodesic metric space. We say a path $\gamma : I \to X$ is a (K,δ,s)-quasi-geodesic if the inequality $length_s(\gamma[x,y]) \leq K d_X(\gamma(x),\gamma(y)) + \delta$ holds for any $x,y \in I$, where $K \geq 1, \delta, s \geq 0$ are fixed constants, and $length_s$ for $s > 0$ is "arclength on the scale s" defined by $length_s(\gamma[x,y]) = sn$, where n is the smallest number for which $[x,y]$ can be subdivided into n closed subintervals $J_1, \cdots, J_n$ with $diam_X(\gamma(J_i)) \leq s$. We say that X has **stability of quasi-geodesics** if for all $K \geq 1, \delta, s \geq 0$, there exists $R > 0$ such that any (K,δ,s)-quasi-geodesic $\alpha : I \to X$ with endpoints x,y remains in the $R-$neighborhood of any geodesic $[xy]$. □

Lemma 3.8. *If X has a coarsely transitive path family Γ with the contraction property, then X has stability of quasi-geodesics. In addition, the paths in Γ themselves are uniform quasi-geodesics.*

Proof. It is Lemma 6.1 of [13]. The idea of this proof is quoted as follows. Assume γ is transitive. Choose $\gamma : [0,M] \to X$ in Γ and α is a (K,δ,s)-quasi-geodesic with the same endpoints with γ. Note that a $(K,\delta,0)$-quasi-geodesic is also a $(K,\delta+s,s)$-quasi-geodesic, so we may assume $s > 0$. To show α remains in a $R(K,\delta,s)$-neighborhood of γ, we obeserve that large excursions of α away from γ can be circumvented by short cuts that travel along the projections to γ, using the contraction property. The projection from α to γ means for any $t \in [0,M]$, we can find $u \in [0,L]$ such that $d(\gamma(t),\gamma(\pi(\alpha(u)))$ is bounded by a uniform constant, realized by chopping α into bounded-length pieces and applying conditions (1) and (2) of the contraction property. Then apply this projection to a geodesic α and a quasi-geodesic β with the same endpoints, first projecting β to a path γ and then projecting γ to α, in which way we obtain a uniform bound. □

Lemma 3.9. *Stability of quasi-geodesics implies hyperbolicity.*

Proof. It is Lemma 6.2 of [13]. Just verify the thin triangle condition. Let $x,y,z \in X$, $z' \in [xy]$ is a point that minimizes the distance from z to $[xy]$. Claim that the broken geodesic $[xz'] \cup [z'z]$ is a (3,0,0)-quasi-geodesic. (This is because assuming $z' \neq x$, letting $u \in [xz']$ and $v \in [z'z]$, then $d(u,v) \geq d(z',v')$, also $d(u,v) \geq d(u,z') - d(z',v)$, then we get $3d(u,v) > d(z',v) + d(u,v')$.) Then by the definition of stability of quasi-geodesics, $[xz'] \cup [z'z]$ is in a uniform δ-neighborhood of $[xz]$, and particularly $[xz']$ is. Apply this to y replacing x and we see the thin triangle condition is satisfied. □

If we combine above two lemmas, the theorem initially stated in this step is proved. Thus the part for hyperbolicity is done.

Step 4: The last step is verifying the curve complex has infinite diameter except $S = S_{0,p}$ with $p \leq 3$, and it is explicitly implied by the proposition beneath. It's worth mentioning infinite diameter, since from the definition of Gromov hyperbolicity, a metric space with finite diameter is obviously Gromov hyperbolic.

Proposition 3.10. *For a non-sporadic surface S there exists $c > 0$ such that for any pseudo-Anosov $h \in Mod(S)$, any $\gamma \in C_0(S)$ and any $n \in \mathbb{Z}$, $d_C(h^n(\gamma),\gamma) \geq c|n|$.*

Proof. It is Proposition 3.6 of [13]. First of all, the definition of a pesudo-Anosov map is a homeomorphism $h : S \to S$ of a closed surface S such that there exists a transverse pair of measured foliations on S, F^s (stable) and F^u (unstable), and a real number $\lambda > 1$ such that the foliations are preserved by f and their transverse measures are multiplied by $1/\lambda$ and λ. The number λ is called the stretch factor or dilatation of f. The key idea in this proof is that, in view of Lemma 3.4 above, if we can construct a sequence $\tau_0, \tau_1, \cdots, \tau_n$ of train-tracks by induction, such that $int(PE(\tau_{j+1}) \subset int(PE(\tau_j)$, then if an essential curve $\beta \in C_0(S)$ is not carried on any diagonal extension of τ_0 and α is carried on some diagonal extension of τ_{n-1}, then $d_C(\alpha,\beta) \geq n$. The construction is derived from how h acts on a train-track τ_0 where h is in a standard form such that $h(\tau_0)$ is carried on τ_0 and fills it. By analyzing matrices representation through action on the branch set, we get that given any $\delta > 0$, there exists m_1 depending only on δ and S, such that for some $m \leq m_1$, $max_{b \in \mathscr{B} \setminus \mathscr{B}_0} h^m(x)(b) \leq \delta min_{b \in \mathscr{B}_0} h^m(x)(b)$ for any $x \in P(\tau_0)$. Then $h^m(PE(\tau_0)) \subset int(PE(\tau_0))$. Then the sequence is constructed by letting $\tau_j = h^{mj}(\tau_0)$, and by induction we find $PE(\tau_{j+1}) \subset int(PE(\tau_j))$. Applying the above idea, we know if $\beta \in C_0(S), \beta \notin PE(\tau_0)$ and $h^m(\beta) \in PE(\tau_0)$, then $h^{nm}(\beta) \in PE(\tau_{n-1})$, so $d_C(h^{nm}(\beta),\beta) \geq n$. Next, for arbitrary $n \in \mathbb{Z}$, $|n| \leq d_C(h^n m(\beta),\beta) \leq m \cdot d_C(h^n(\beta),\beta)$. Thus $d_C(h^n(\beta),\beta) \geq |n|/m$. Let $c = 1/m_1$.

□

With these 4 steps, the proof is finished.

□

4 Improvement of the result

In the latter section, I already gave a brief introduction to the proof given by Masur and Minsky of the Gromov hyperbolicity of the curve complex. The proof is a canonical one using the relation between the geometry of the Teichmüller space to the curve complex, the treatment of train-tracks, etc. Still, there is space for improvement, as said by the authors. On the one hand, Masur and Minsky looked forward to a purely combinatorial proof which does not use Teichmüller theory; on the other hand, they hoped to find a proof which can give an effective estimate of the constant δ in the definition of the Gromov hyperbolicity, for instance, to place some bounds on δ, which their proof did not provide since it depends on bounds obtained from a compactness argument in the Moduli space. Thus in this section I will introduce some later work aimed at the two aspects for improvement.

Bowditch gave a more combinatorial proof in [4] in 2006. His proof is logically independent of the proof given by Masur and Minsky, although some ideas are inspried by the ideas in Masur and Minsky's proof, such as the study of nested train-tracks and the geometry of the Teichmüller geodesics, but Bowditch phrased them more combinatorially in terms of the intersection number. The key alternation of Bowditch's proof is avoiding the step in [13] of verifying the contraction property of a family of paths in $C(S)$, that is, avoiding constructing a uniform contraction from $C_0(S)$ onto a path using a sophisticated analysis of nested train-tracks. Instead, Bowditch directly showed that any triangle formed by three paths in the curve complex is "thin" in an appropriate sense. For $\alpha,\beta,\gamma \in C_0(S)$, he uses a notion called the "contre" denoted as $\phi(\alpha,\beta,\gamma) \in C_0(S)$, which is defined to be a curve $\delta \in C_0(S)$ with $i(\bar{\alpha},\delta) \leq R, i(\bar{\beta},\delta) \leq R$ and $i(\bar{\gamma},\delta) \leq R$, where $\bar{\alpha},\bar{\beta},\bar{\gamma}$ are normalizations of α,β,γ by assigning weights to them such that $i(\bar{\alpha},\bar{\beta}) = i(\bar{\beta},\bar{\gamma}) = i(\bar{\gamma},\bar{\alpha}) = 1$, and R is a given constant. Then the key point is to show that if $\gamma,\delta \in C_0(S)$ are adjacent, then $d_C(\phi(\alpha,\beta,\gamma),\phi(\alpha,\beta,\gamma))$ is bounded (which is stated similarly in [13] but proved in a different way), where the author does a similar construction for multicurves, and I refers to Proposition 4.11 in [4] for details. With this fact, one sees that the paths in $C(S)$ are uniformly quasigeodesics (Lemma 3.3 in [4]). Then the Gromov hyperbolicity of the curve comlex follows from the proposition below, where hyperbolicity is characterized by a subquadratic isoperimetric inequality. The subquadratic isoperimetric inequality means that for a given $M \geq 0$, every loop in the 1-skeleton bounds a spanning disc of mesh at most M and area at most $o(length(\gamma)^2)$, where $\gamma : S^1 \to C_1(S)$ is the boundary of a unit disc D, a cellulation of D is a representation of D as a CW-complex, a spanning disk for γ is an extension of γ to the 1-skeleton of some cellulation of D, and its mesh is the maximal length of the boundary of a 2-cell of the cellulation, its area is the number of 2-cells.

Proposition 4.1 (Bowditch)**.** *To each pair $a,b \in C_0(S)$, we define a line $(\Lambda_{ab},\leq_{ab})$ from a to b to be a subset $\Lambda_{ab} \subset X$ with a coarse order (relexive, transitive and satisfies the dichotomy rule, but not necessarily antisymmetric). For any $a,b \in C_0(S)$, we suppose $\Lambda_{ab} = \Lambda_{ba}$, and given $x,y \in \Lambda$ with $x \leq_{ab} y$, $\Lambda_{ab}[x,y] = \Lambda[y,x] = \{z \in \Lambda_{ab} | x \leq_{ab} z \leq_{ab} y\}$. For the contres, suppose $\phi(a,b,c) = \phi(b,c,a) = \phi(c,a,b)$, and that $\phi(a,a,b) = a$, and $\phi(a,b,c) \in \Lambda_{ab} \cap \Lambda_{bc} \cap \Lambda_{ca}$. If further there is a constant $K \geq 0$ with the following properties, then $C(S)$ is hyperbolic with hyperbolicity constant depending only on K.*

(1) If $a,b,c \in C_0(S)$, then $HausDist(\Lambda_{ab}[a,\phi(a,b,c)],\Lambda_{ac}[a,\phi(a,b,c)]) \leq K$, where HauDist is the Hausdorff distance with respect to the metric d_C.

(2) If $x, y \in C_0(S)$ with $d_C(x,y) \leq 1$, then diam$\Lambda_{ab}[\phi(a,b,x), \phi(a,b,y)] \leq K$.

(3) If $c \in \Lambda_{ab}$, then $\Lambda[c, \phi(a,b,c)] \leq K$.

Bowditch also showed that the hyperbolicity constant for $C_1(S)$ is bounded by a logarithmic funtion of complexity, stated as next proposition which is Proposition 6.1 in [4]. He remarked that it's not clear what is the best estimate. Notice that the bound he gave is funtional but not numerical.

Proposition 4.2. *There is a function $k : \mathbb{N} \to \mathbb{N}$ with $k(n) = O(log(n))$, so that the 1-skeleton of the curve complex of a surface of genus g with p punctures, is $k(3g+p-4)$-hyperbolic, provided $3g+p-4 > 0$.*

Besides, Bowditch generalizes the result for the curve complex starting from variations on the definition of the curve complex. First, for any given $m \geq 0$, he writes $\mathscr{G}_m = \mathscr{G}_m(S_{g,p})$ for the graph with the same vertex set $C_0(S)$, where $\alpha, \beta \in C_0(S)$ are defined to be adjacent if $i(\alpha, \beta) \leq m$. If $3g+p-4 > 0$, $\mathscr{G}_m$ is hyperbolic, too, since the embedding of $C(S)$ in $\mathscr{G}_m$ is quasi-isometry. Remark that for $g = 1, p \leq 1, m = 1$ or $g = 0, p = 4, m = 2$, $\mathscr{G}_m$ is the Farey gragh whose hyperbolicity have been talked about before. A further generaliztion begins from changing the vertex set to X_n, the set of curves with self-intersection number at most n, given $n \geq 0$. Then if $m \geq n$,the gragh $\mathscr{G}_{m,n}$ is the graph with vertices $\alpha, \beta \in X_n$ adjacent if $i(\alpha, \beta) \leq m$. Not only $\mathscr{G}_m$ embeds in $\mathscr{G}_{m,n}$ as a full graph, but also the inclusion of $\mathscr{G}_m$ into $\mathscr{G}_{m,n}$ is quasi-isometry. Thus $\mathscr{G}_{m,n}$ is hyperbolic whenever $m > n \geq 1$ or $m \geq n \geq 2$. Another generalization (see also [9]) is adding to the vertex set the classes of arcs with endpoints in the boundary components but not punctures. Two arcs are in the same class if one can be deformed to the other through such arcs, and two vertices are adjacent if they can be realized disjointly. With the curve complex embedded by a quasi-isometry, this new graph is also hyperbolic.

After Bowditch gave a logarithmic functional bound for the hyperbolicity contant, Aougab showed that there exists a uniform hyperbolicity constant independent of the topological charateristic of the surface, in [1] in 2013. Also notice that the bound is for $C_1(S)$, and although with every simplex endowed with a standard metric, the full complex is quasi-isometric to its 1-skeleton, so the property of being Gromov hyperbolic is invariant, but the specific constant δ is not. Thus the result of Aougab did not give a uniform bound for the hyperbolicity constant of $C(S)$. Note that independent proof of this result is also given by Bowditch[5], by Hensel, Przytycki, and Webb[19], and by Clay, Rafi, and Schleimer[17].

Theorem 4.3. *There exists $k > 0$ so that for any admissible choice of g, p, $C_1(S_{g,p})$ is $k-$hyperbolic.*

Aougab's proof inherited the notions and notations of Bowditch discussed just now, and his main tool is the following theorem:

Theorem 4.4 (Auogab)**.** *For each $\lambda \in (0,1)$, there is some $N = N(\lambda) \in \mathbb{N}$ such that if $\alpha, \beta \in C_0(S_{g,p})$, whenever $3g+p-4 > N$ and $d_C(\alpha, \beta) \geq k$, we have $i(\alpha, \beta) \geq (\frac{(3g+p-4)^\lambda}{f(3g+p-4)})^{k-2}$, where $f(3g+p-4) = O(log_2(3g+p-4))$.*

Both Bowditch's proof and Aougab's make use of Riemannian geometry. The proofs in [19] given by Hensel, Przytycki and Webb and in [17] given by Clay, Rafi and Schleimer are both combinatorial in nature. Aougab used this result to show uniform boundedness of the curve complex distance between two vertex cycles of the same train track, and of the Lipschitz constants of the map from Teichmüller space to $C(S)$ sending a Riemann surface to the curves of shortest extremal length. Bowditch used this result to give a more concrete criterion of hyperbolicity. Both [17] and [19] talked about the hyperbolicity of the complex of curves and arcs defined above.

As for the expicit numerical bound, Bowditch showed that if $2g+p \geq 107$, then $C_1(S_{g,p})$ is 1780-hyperbolic; if $2g+p \geq 14$, then $C_1(S_{g,p})$ is 2492-hyperbolic; if $g \geq 8$, $C_1(S_{g,0})$ is 1780-hyperbolic, etc. In Hensel, Przytycki, and Webb's artical, the argument below seems to give the optimal constants: 17 for the curve complex, 7 for the arc complex, though the definition for hyperbolicity is in another sense.

Theorem 4.5 (Hensel,Przytycki and Webb)**.** *If $C(S)$ is connected, then it is 17-hyperbolic, in the sense that, for every geodesic triangle, there is a vertex a distance at most 17 from each of its sides.*

5 Applications

In this section, I will introduce the applications of the Gormov hyperbolicity of the curve complex in Teichmüller spaces and mapping class groups.

5.1 To Teichmüller space

Teichmüller being a geodesic metric space, Masur and Wolf have shown that Teichmüller space with the Teichmüller metric is not Gormov hyperbolic, discussed before in Section 2.3, due to the existence of infinite-diameter regions $\{H_\alpha\}_{\alpha \in C_0(S)}$, where $H_\alpha = \{x \in T(S) : Ext_x(\alpha) < \varepsilon_0\}$ for sufficiently small ε_0. However, one can observe that due to the Collar Lemma (a classical result due to Keen [14]), s set of curves $\alpha_1, \cdots, \alpha_k$ spans a simplex in $C(S)$ if and only if $H_{\alpha_1} \cap \cdots H_{\alpha_k}$ is non-empty, thus the intersection pattern can be fully interpreted in $C(S)$, which is Gromov hyperbolic. Recalling the definition of relatively hyperbolicity given by Farb in Section 2.2, the constrcuction of electric space and electric distance actually cares about the intersection pattern of these regions, by shrinking the inner geometric structure of them. It is interpreted as the following lemma:

Lemma 5.1 (Masur, Minsky)**.** *The electric space $(\hat{T}(S), d_e)$ defined with respect to the family $\{H_\alpha\}_{\alpha \in C_0(S)}$, is quasi-geometric to $C_1(S)$.*

Then due to hyperbolicity of the curve complex, the relatively hyperbolicity of the Teichmüller space follows naturally:

Theorem 5.2 (Masur, Minsky)**.** *The Teichmüller space $T(S)$ is relatively hyperbolic with respect to the family of regions $\{H_\alpha\}_{\alpha \in C_0(S)}$.*

5.2 To mapping class group

The mapping class group of a surface S denoted as Mod(S) is the group of orientation-preserving self-homeomorphims of S module isotopy. Mod(S) has a natural action on the curve complex $C(S)$ by mapping $[\alpha] \in C_0(S)$ to $[\phi(\alpha)]$ for each $\phi \in$ Mod(S). Thus the geometry and combinatorics of $C(S)$ can be applied to study group theoretic properties of Mod(S).

Recall that a finitely generated group is called Gromov hyperbolic if its Caylay graph with respect to some finite generating set with the word metric is δ-hyperbolic for some δ. A classical result of Dehn showed that Mod(S) is finitely generated, and both Lickorish and Humphries gave explicit generators respectively, see Chapter 4 of [2]. However, it is well-known that mapping class groups are usually not Gormov hyperbolic since it contains abelian subgroups of high rank. Due to the same intuition as in the Teichmüller space, if we can somehow "shrink" these abelian subgroups in the Caylay graph $\mathscr{G}$ of Mod(S), the remaining information is encoded by how the mapping class group acts on the curve complex:

Lemma 5.3 (Masur, Minsky)**.** *Fixing a choice of generating set* Γ *and representatives* $\{\alpha_1, \cdots, \alpha_N\}$ *of* $Mod(S)$*-orbits, the family of subgroups* $\{G_\beta\}_{\beta \in C_0(S)}$ *is defined by* $G_\beta = \{g \in Mod(S) | g(\alpha_i) = \beta\}$ *where* α_i *is the unique representative of* β *in its* $Mod(S)$*-orbit. Then the coned-off Caylay graph* $(\hat{\mathscr{G}}, d_e)$ *of* $Mod(S)$ *with respect to* $\{G_\beta\}_{\beta \in C_0(S)}$ *is quasi-isometric to* $C_1(S)$*.*

Similarly, the hyperbolicity of the curve complex leads to the relatively hyperbolicity of the mapping class group.

Theorem 5.4 (Masur, Minsky)**.** $Mod(S)$ *is relatively hyperbolic with respect to* $\{G_\beta\}_{\beta \in C_0(S)}$*.*

However, notice that the result of hyperbolicity of the curve complex is not "proper", since we know the curve complex is not locally finite. This makes its applications a bit difficult since unlike proper cases, the union of the curve complex with its boundary at infinity is not compact in a natural topology. Note that this problem is nicely resolved in the case of the curve complex for the torus, where we have an explicit description of geodesics in spite of the fact that the link of every vertex is infinite. For general cases, researchers thought of various ways to overcome this difficulty as follows.

In [9], Masur and Minsky aimed at developing tools to apply the theory of hyperbolic spaces and groups to algorithmic questions for the mpping class group. They introduced the notion of a hierarchy of tighted geodesics, which is a combinatorial tool to tie together the hyperbolic levels of a layered structure of $C(S)$. This structure is induced by a family of subsurface projection maps, in analogy to closest-point projections to horoballs in classical hyperbolic, showing how the geometry of the links of vertices in $C(S)$ is tied to the geometry of $C(S)$. Using these constructions, they derived a number of properties of $C(S)$ similar to those of locally finite complexes, such as a finiteness result for geodesics with given endpoints (Theorem 6.14 in [9]), and a convergence criterion for sequences of geodesics (Theorem 6.13 in [9]). These properties newly derived were then used to get a linear bound on the length of the shortest word conjugating two pseudo-Anosov mapping classes, stated as follows.

Theorem 5.5 (Masur, Minsky)**.** *Fix a surface S of finite type and a generating set for* $Mod(S)$*. If* h_1, h_2 *are words describing conjugate pseudo-Anosov elements, then the shortest conjugating element w has word length* $|w| \leq C(|h_1| + |h_2|)$*, where the constant C depends only on S and the generating set.*

In [6], Dahmani, Guirardel, and Osin tried to suggest a general approach to study hyperbolic and relatively hyperbolic groups. They suggested two ways: one is the notion of a hyperbolically embedded collection of subgroups, as a generalization of the peripheral structure of relatively hyperbolic groups; the other is the notion of very rotating families of subgroups which provide a suitable framework to study collections of subgroups satisfying small cancellation conditions. They applied these techniques to the action of mapping class groups on the curve complex, for example, based on their hyperbolicity, and got the following result as Theorem 2.19 in [6]:

Theorem 5.6 (Dahmani, Guirardel, Osin)**.** *Let S be a (possibly punctured) orientable closed surface, and let $Mod(S)$ be its mapping class groups.*

(1) For every pseudo-Anosov element $a \in Mod(S)$, $E(a)$ hyperbolically embeds in $Mod(S)$, where $E(a)$ is the maximal virtually cyclic subgroups containing a.

(2) For every $\alpha > 0$, there exists $n > 0$ such that for every pseudo-Anosov element $a \in Mod(S)$, the cyclic subgroup $\langle a^n \rangle$ is α-rotating.

(3) Every subgroup of $Mod(S)$ either is virtually abelian or virtually surjects onto a group with a non-degenerate hyperbolically embedded subgroup.

I refer to [6] for specific definitions of hyperbolically embedding subgroups and $\alpha-$rotating subgroups, due to limited space. More importantly, Dahmani, Guirardel, and Osin used the teichniques they introduced to successfully solve two open questions in mapping class groups.

Problem 5.7 (Problem 2.12(A) in Kirby's list)**.** *Let S be a closed orientable surface. Does $Mod(S)$ have a non-trivial purely pseudo-Anosov normal subgroup? (a subgroup of a mapping class group is called purely pseudo-Anosov if all its non-trivial elements are pseudo-Anosov.)*

Problem 5.8 (Ivanov)**.** *Let S be a closed orientable surface. Is the normal closure of a certain nontrivial power of a pseudo-Anosov element of $Mod(S)$ free?*

Dahmani, Guirardel, and Osin gave both problems positive answers in Theorem 2.31 of [6].

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